REFERENCES

1. A. Okubo, *Diffusion and Ecological Problems: Mathematical Models*, pp. 166–168. Springer, Berlin (1980).
2. P. Frank und R. v. Mises, *Die Differential- und Integralgleichungen der Mechanik und Physik*, p. 578. Vieweg, Braunschweig (1935).
3. J. Unnam and D. R. Tenney, Effect of zone size on the convergence of exact solutions for diffusion in single phase systems with planar, cylindrical or spherical geometry, *Metall. Trans. A.* **12A**, 1369–1372 (1981).
4. M. Sato, Presentation of a new formulation of negentropy (1st Report, Basic concepts), *Bull. J.S.M.E.* **25**, 599–605 (1982).
5. M. Sato, Presentation of a new formulation of negentropy (2nd Report, Description of work by negentropy), *Bull. J.S.M.E.* **25**, 1551–1558 (1982).

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AN APPROXIMATE ANALYTICAL SOLUTION TO THE FREEZING PROBLEM SUBJECT TO CONVECTIVE COOLING AND WITH ARBITRARY INITIAL LIQUID TEMPERATURES

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NOMENCLATURE

- c heat capacity
- h heat transfer coefficient
- k thermal conductivity
- L latent heat of fusion
- Ste Stefan number, $C_s(T_m - T_a)/L$
- Ste' modified Stefan number, $C_f(T_0 - T_m)/L$
- t time
- T temperature
- V parametric variable
- y space coordinate
- Y dimensionless distance, hy/k_f

Greek symbols

- α thermal diffusivity
- ε solid thickness
- η dimensionless solid thickness, $h\varepsilon/k_s$
- θ dimensionless temperature, $(T - T_m)/(T_0 - T_m)$
- ρ density
- τ dimensionless time, $\alpha_f h^2 t/k_s^2$

Subscripts

- a ambient
- f liquid phase
- m freezing
- 0 initial
- s solid phase
- t surface

1. INTRODUCTION

THE ONE-DIMENSIONAL moving boundary problems associated with freezing and melting have always been of great interest to engineers and scientists. The problem has wide application, e.g. in freezing and melting of lake ice, cooling of large masses of igneous rock, solidification of castings and purification of materials.

The problem is characterized by the existence of a moving boundary resulting from phase change. Analytical solutions are possible only for a few special classes of boundary and initial conditions, for example, Stefan or Neumann's problems [1]. For other more complicated boundary conditions, different assumptions have to be used. Series solutions have been attempted [2–5]. More recently, Foss [6] presented a simple approximate solution to an important class of moving boundary problems; the freezing and melting of lake ice. A convective boundary was applied to the air–ice interface. The solution compared closely with Westphal's more accurate series solution [5]. However, the initial water temperature in both cases was assumed to be at the fusion temperature, thus, ignoring conduction in the liquid phase. This limits the usefulness of the solutions in many practical applications.

This paper presents an approximate analytical solution to the moving boundary problem associated with the freezing of a semi-infinite phase-change medium due to convective cooling in the fixed boundary and with an initial temperature which can be higher than the fusion temperature.

2. STATEMENT OF THE PROBLEM

The problem considers a semi-infinite body of phase change medium extending from $y = 0$ to ∞. The initial temperature of the liquid (T_0) is assumed to be uniform and higher than the fusion temperature (T_m). Convective cooling is applied to the fixed boundary ($y = 0$) at $t \geqslant 0$ with a constant heat transfer coefficient (h) and a constant sub-freezing ambient temperature (T_a) (Fig. 1). The problem can be divided into two: before and after freezing at the free surface.

Before freezing ($t < t_m$), the problem is relatively simple and

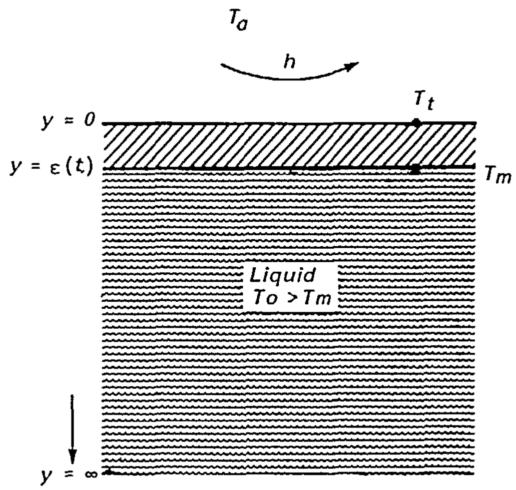

FIG. 1. System geometry.

an analytical solution is available [7]

$$\frac{T_f(y,t)-T_0}{T_a-T_0} = 1-\mathrm{erf}(y^2/4\alpha_f t)^{1/2} - \left[\exp\left(\frac{hy}{k_f}+\frac{h^2\alpha_f t}{k_f^2}\right)\right] \times \{1-\mathrm{erf}(y^2/4\alpha_f t)^{1/2}+[(h^2\alpha_f t/k_f^2)^{1/2}]\}. \quad (1)$$

The problem becomes more complicated after a solid layer forms at the surface because of the existence of two phases (solid and liquid), a fixed and a moving boundary. There are two transient heat conduction equations, one for each of the two phases

$$\frac{\partial T_s}{\partial t} = \alpha_s \frac{\partial^2 T_s}{\partial y^2}, \quad 0 \leqslant y \leqslant \varepsilon(t) \quad (2)$$

and

$$\frac{\partial T_f}{\partial t} = \alpha_f \frac{\partial^2 T_f}{\partial y^2}, \quad \varepsilon(t) \leqslant y \leqslant \infty. \quad (3)$$

The boundary conditions are:

(i) at the fixed boundary ($y = 0$)

$$k_s \frac{\partial T_s}{\partial y} = h(T_t - T_a) \quad (4)$$

and

(ii) at the moving boundary [$y = \varepsilon(t)$]

$$T_s = T_f = T_m, \quad (5)$$

$$\rho_s L \frac{\partial \varepsilon}{\partial t} = k_s \frac{\partial T_s}{\partial y} - k_f \frac{\partial T_f}{\partial y}. \quad (6)$$

Equation (6) gives the rate of freezing and is determined by the difference between the rate of heat conduction into the solid phase and the rate of heat conduction from the liquid phase at the interface.

The initial conditions are

$$\varepsilon(t_m) = 0 \quad (7)$$

and

$$T_f = T_f(y, t_m) \quad (8)$$

where $T_f(y, t_m)$ is given by equation (1).

To solve the set of equations [equations (2)–(6)] analytically can be complicated [4]. However, by making some physically realistic assumptions, a simple approximate solution can be obtained. This is detailed in the next section.

3. SOLUTION

The approximations are with regard to the temperature distributions in the two phases:

(1) When the movement of the interface is slow, which is true in most cases, quasi-steady-state heat conduction in the solid phase can be assumed. The temperature distribution in the solid phase is thus linear accordingly. The rate of heat conduction into the solid phase at the interface becomes

$$\frac{\partial T_s}{\partial y} = \frac{T_m - T_a}{\varepsilon(t) + k_s/h}. \quad (9)$$

(2) With the formation of a solid layer at the surface, the boundary condition for the liquid body changes from convective at $y = 0$ to isothermal ($T = T_m$) at $y = \varepsilon(t)$. If the interface is stationary and the liquid is at a uniform initial temperature, an analytical solution exists [1]. The temperature gradient at the interface is given by

$$\frac{\partial T_f}{\partial y} = \frac{T_0 - T_m}{(\pi\alpha_f(t-t_m))^{1/2}}. \quad (10)$$

Since the fluid has been cooled by convection before freezing occurs at the surface, its temperature distribution at time of freezing may be approximated by assuming the fluid had always been cooled with an isothermal boundary at the fusion temperature for a different length of time, say t^*. This time can be found by equating the temperature gradient governed by the convective boundary at t_m to the gradient obtained using an isothermal boundary, i.e.

$$h(T_m - T_a) = k_f \frac{T_0 - T_m}{(\pi\alpha_f t^*)^{1/2}}. \quad (11)$$

Equation (11) ensures the continuity of the liquid temperature gradient at the surface at time of freezing when different solutions are used. Temperature distributions, however, are not necessarily the same, although they turn out to be quite close in the cases considered. This will be discussed later.

Solving for t^*, we have

$$t^* = \frac{1}{\pi\alpha_f}\left(\frac{k_f}{h}\right)^2\left(\frac{T_0 - T_m}{T_m - T_a}\right)^2. \quad (12)$$

Substituting equations (9) and (10) into equation (6) and taking t^* into account, we have

$$\rho_s L \frac{\partial \varepsilon}{\partial t} = k_s \frac{T_m - T_a}{\varepsilon(t) + k_s/h} - k_f \frac{T_0 - T_m}{[\pi\alpha_f(t - t_m + t^*)]^{1/2}}. \quad (13)$$

This is the equation which governs the rate of freezing for $t > t_m$.

Introducing the following dimensionless variables:

$$\eta = \frac{h}{k_s}\varepsilon \quad \text{and} \quad \tau = \left(\frac{h}{k_s}\right)^2 \alpha_f t. \quad (14)$$

Equation (13) can be written in a non-dimensional form

$$\frac{\partial \eta}{\partial \tau} = \left(\frac{\alpha_s}{\alpha_f}\right)\frac{Ste}{\eta + 1} - \left(\frac{\rho_f}{\rho_s}\right)\frac{Ste'}{[\pi(\tau - \tau_m + \tau^*)]^{1/2}} \quad (15)$$

with

$$\tau^* = \frac{1}{\pi}\left(\frac{\rho_f \alpha_f Ste'}{\rho_s \alpha_s Ste}\right)^2. \quad (16)$$

The solution to equation (15) in parametric form is [8]

$$\eta = V\{\tau^* \exp[I(V_m) - I(V)]\}^{1/2} - 1, \quad (17a)$$

$$\tau = \tau^*\{\exp[I(V_m) - I(V)] - 1\} + \tau_m \quad (17b)$$

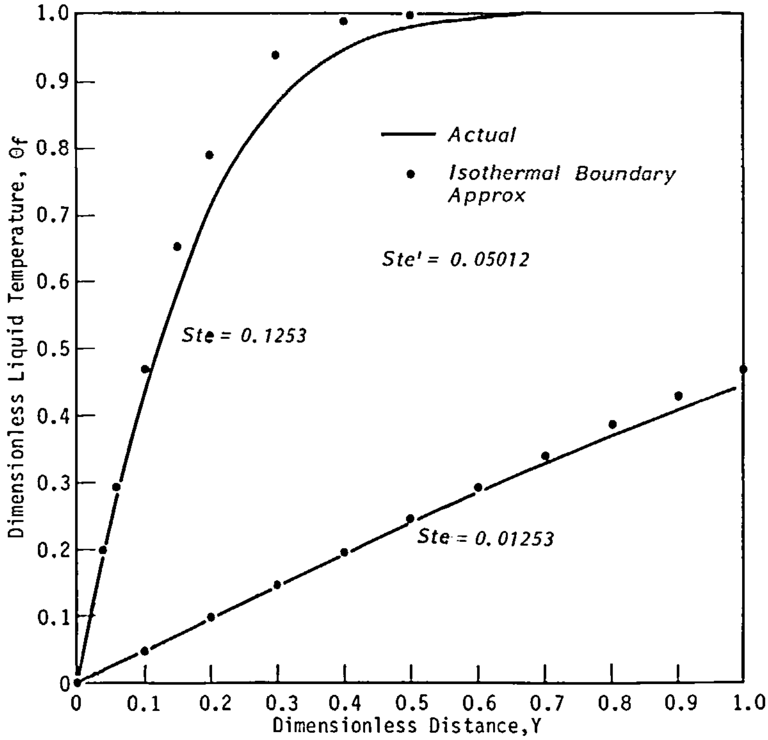

FIG. 2. Temperature distribution at time of freezing.

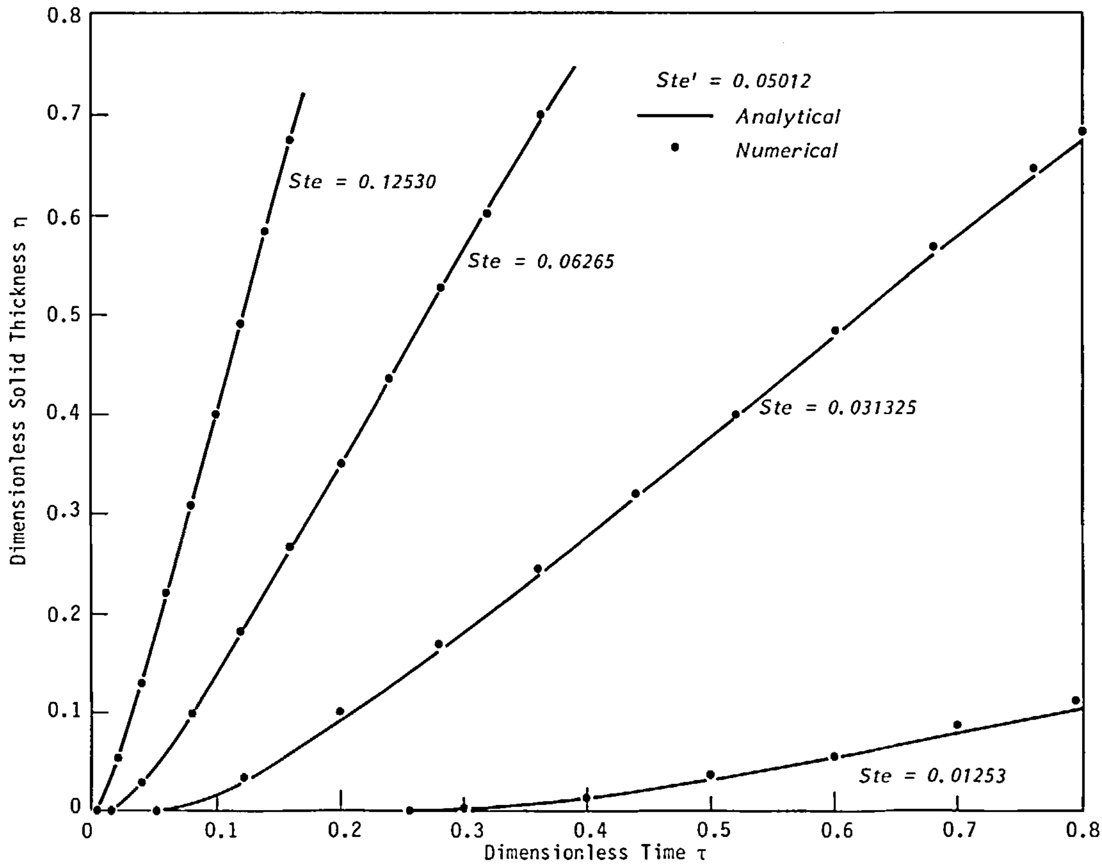

FIG. 3. Thickness of solid material vs time.

where

$$I(V) = \ln(\tfrac{1}{2}V^2 + bV - a) + \frac{b}{(b^2+2a)^{1/2}} \ln\left[\frac{V+b+(b^2+2a)^{1/2}}{V+b-(b^2+2a)^{1/2}}\right] \quad (18)$$

and

$$V_m = 1/(\tau^{*1/2}). \quad (19)$$

4. DISCUSSION

The most important approximation used in the present derivation is the assumption that the liquid temperature distribution governed by convective cooling at time of freezing can be replaced by the temperature profile obtained using an isothermal boundary at $t = t^*$. t^* is obtained such that the liquid temperature gradient at the surface remains the same. Thus, the initial rate of solidification will not be affected. To examine the validity of the assumption used, the two liquid temperature profiles are compared. This is shown in Fig. 2 for $Ste' = 0.05012$ and $Ste = 0.01253$ and 0.1253. For water, this corresponds to $T_0 = 4°C$ and $T_a = -2°C$ and $-20°C$, respectively. The actual or convectively cooled profiles are obtained from equation (1) with $t = t_m$. The isothermally cooled profiles are given by [1]

$$\frac{T_f(y,t^*) - T_m}{T_0 - T_m} = \text{erf}\,\frac{y}{2(\alpha_f t^*)^{1/2}}. \quad (20)$$

It can be seen that the corresponding temperature profiles compare very well with each other. Thus, the use of the profile obtained using an isothermal boundary at t^* to replace the actual temperature profile at t_m is indeed a good approximation and its effect on surface freezing rate is expected to be small in general.

Typical results obtained by using the present approximate solution [i.e. equation (17)] are plotted in Fig. 3 for $Ste' = 0.05012$. Four Stefan numbers, $Ste = 0.01253$, 0.031325, 0.06265 and 0.1253, are used. This corresponds to $T_a = -2°C$, $-5°C$, $-10°C$ and $-20°C$ respectively for water. The freezing times (τ_m) for the curves were obtained using equation (1) with $T_f(y=0) = T_m$ and $y = 0$.

As expected, the results show that the freezing rate increases rapidly as Ste increases, i.e. with lower ambient temperatures. Physically, the Stefan number is the ratio of the driving force for freezing and the resistance to freezing. Thus, higher Stefan numbers result in faster rates of freezing.

Also shown in Fig. 3 are results obtained numerically. The numerical results were obtained using as many as 160 grid points for accuracy. Details of the numerical procedure can be found elsewhere [8]. It can be seen that the analytical and numerical results compare very well, especially for higher Stefan numbers. For lower Stefan numbers, the analytical solution predicts freezing rates slightly lower than the numerical solution.

It should also be noted that: (i) the approximate solution obtained [equation (17)] is in dimensionless form. Therefore, the results shown in Fig. 3 are applicable for different phase change media and under different sets of boundary and initial conditions; (ii) for the special case with $T_0 \to T_m$, the present solution reduces identically to Foss's solution [8].

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids* (2nd edn.), pp. 282–296. Clarendon Press, Oxford (1959).
2. F. Jackson, The solution of problems involving the melting and freezing of finite slabs by a method due to Portnov, *Proc. Edinb. Math. Soc.* **14**, 109–128 (1964).
3. D. Langford, New analytical solutions of the one-dimensional heat equation for temperature and heat flow rate both prescribed at the same fixed boundary (with application to the phase change problem), *Q. Appl. Math.* **24**, 315–322 (1966).
4. B. Boley, A general starting solution for melting and solidifying slabs, *Int. J. Engng Sci.* **6**, 89–111 (1968).
5. K. O. Westphal, Series solution of freezing problem with the fixed surface radiating into a medium of arbitrary varying temperatures, *Int. J. Heat Mass Transfer* **10**, 195–205 (1967).
6. S. D. Foss, An approximate solution to the moving boundary problem associated with the freezing and melting of lake ice, *A.I.Ch.E. Symp. Ser.* **74** (174), 250–255 (1978).
7. P. J. Schneider, *Conduction Heat Transfer*. Addison-Wesley, Cambridge (1955).
8. A. M. C. Chan and M. Shoukri, On the analysis of water cooling and freezing in the CANDU vacuum building due to environmental conditions, Ontario Hydro Research Division Report (1983).

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OPTICAL ILLUSTRATION OF LIQUID PENETRATION TO THE VAPOUR FILM IN INVERTED ANNULAR BOILING

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1. INTRODUCTION

Inverted annular film boiling is an important stage in the reflooding phase of a loss-of-coolant accident (LOCA) in light water reactors. Heat transfer coefficients for inverted annular flow of water in a single vertical tube have been measured in steady-state experiments [1] and for more realistic flow conditions [2–4]. Seban *et al.* [5] and Edelman [4] have examined the experimental results in the region immediately downstream of the quench front for cases in which the water in this region was subcooled or had a very low quality and suggested that existing models for film boiling are inadequate for the specification of the heat transfer coefficient. Unfortunately models based on laminar [6] or turbulent [7] vapour boundary layers underpredict both the heat transfer coefficient and the vapour film thickness in the inverted annular region, as they neglect the vapour inlet velocity at the quench front, assuming that all the vapour is generated at the liquid interphase which is considered to be saturated. A